

MHV Amplitudes in $\mathcal{N} = 4$ Super Yang-Mills and Wilson Loops

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Abstract

It is a remarkable fact that MHV amplitudes in maximally supersymmetric Yang-Mills theory at arbitrary loop order can be written as the product of the tree amplitude with the same helicity configuration and a universal, helicity-blind function of the kinematic invariants. In this note we show how for one-loop MHV amplitudes with an arbitrary number of external legs this universal function can be derived using Wilson loops. Our result is in precise agreement with the known expression for the infinite sequence of MHV amplitudes in $\mathcal{N} = 4$ super Yang-Mills. In the four-point case, we are able to reproduce the expression of the amplitude to all orders in the dimensional regularisation parameter ϵ . This prescription disentangles cleanly infrared divergences and finite terms, and leads to an intriguing one-to-one mapping between certain Wilson loop diagrams and (finite) two-mass easy box functions.

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1 Introduction

Even after many years, $\mathcal{N}=4$ supersymmetric Yang-Mills (SYM) remains a fascinating theory that constantly reveals new hidden structures and symmetries. In the last few years, substantial progress has been made in two seemingly unrelated areas of research of the theory.

On the one hand there is the conjecture of Bern, Dixon and Smirnov (BDS) of an exponential formula for planar n -point amplitudes in $\mathcal{N}=4$ super Yang-Mills (SYM) at large N [1]. According to this conjecture, higher-loop amplitudes are determined in terms of the one-loop amplitude together with four constants, which depend on the 't Hooft coupling only. The finite parts of these amplitudes obey a similar exponential formula: the all-loop finite parts are determined purely by the one-loop finite part and two coupling-dependent constants. Furthermore, one of these constants is a well-known physical quantity, known as the cusp anomalous dimension.

In parallel developments, integrability has been used, together with a number of further assumptions, to study the spectrum of gauge invariant operators in $\mathcal{N}=4$ SYM. Among the impressive outcomes of this is the conjectured all-orders formula of [2] for the very same cusp anomalous dimension appearing in the exponentiation formula, thus giving a tantalising potential link between integrability and amplitudes. Calculations of four-point amplitudes using unitarity methods at up to a highly impressive four-loop order have been performed [1, 3], thus determining the cusp anomalous dimension to this order (at least numerically) and leading to an expression for the amplitude in terms of integral functions. This four-loop result was then re-derived in [4], which further confirmed the conjecture of [2]. Using the assumption of pseudo-conformality, originally observed at three [5] and four loops [3], the expressions of the amplitude in terms of the integrals occurring at five loops has been determined [6].

Very recently, Alday and Maldacena have been able to apply for the first time the AdS/CFT correspondence to the calculation of amplitudes, and verified the form of the exponentiation of the four-point amplitude at strong coupling [7]. Remarkably, it turns out that the computation of amplitudes at strong coupling is dual to the computation of the area of a string ending on a lightlike polygonal loop embedded in the boundary of AdS space. This, in turn, is equivalent to the method for computing a lightlike polygonal Wilson loop at strong coupling using AdS/CFT. The edges of the polygon are determined by the external momenta of the amplitude.

In [8] the same Wilson loop (with four lightlike segments) was considered in weakly-coupled gauge theory and it was shown at one loop that it reproduces the

known one-loop four-point amplitude.¹ The infrared divergent pieces come from propagators stretching between adjacent edges, and the finite part of the amplitude comes from propagators stretching between opposite edges.

In this paper we consider Wilson loops around arbitrary lightlike polygons with n sides at one loop, and find precise agreement with one-loop n -point MHV amplitudes in $\mathcal{N}=4$ SYM (divided by the tree amplitude). As for the four-point case, the infrared part is correctly reproduced by considering propagators between adjacent segments and the finite piece is obtained from propagators between non-adjacent segments. This finite part is given by the sum of the (finite) two mass easy box functions in four dimensions with coefficients unity. This ability to separate the divergent and finite parts is a particularly nice feature, and enables us to calculate finite parts purely in four dimensions without the need of a regulator. On the other hand, if we use dimensional regularisation on the finite part as well, we reproduce the all orders in ϵ expression for the two-mass easy box function, and hence the complete all-orders in ϵ expression for the four-point MHV amplitude.

The paper is organised as follows. In section 2 we review the form of n -point MHV amplitudes in terms of two-mass easy box functions, and discuss the BDS conjecture and the strong-coupling calculation of amplitudes as lightlike Wilson loops. In section 3 we present the one-loop calculation of the infinite sequence of MHV amplitudes in $\mathcal{N}=4$ SYM using lightlike Wilson loops. We present our conclusions and comment on the result of our calculations in section 4.

2 One-loop MHV amplitudes in $\mathcal{N}=4$ SYM

In this section we briefly review the expression of the infinite sequence of MHV amplitudes in $\mathcal{N}=4$ SYM at one loop. These amplitudes were determined for the first time in [9] using unitarity and collinear limits. Recently, their expression was re-derived in [10] using one-loop MHV diagrams. In the following discussion we suppress constant factors connected with dimensional regularisation when they are not essential.

The form of the n -point MHV amplitudes at L loops in $\mathcal{N}=4$ SYM is remarkably simple. It turns out that the amplitude is given by the tree-level amplitude, times a

¹More precisely, the calculations of [7] and [8], as well as ours, are insensitive to the polarisations of the particles participating in the scattering. In particular, the tree-level Parke-Taylor amplitude, which appears as a common prefactor in the $\mathcal{N}=4$ MHV amplitudes at any loop order, is not generated by the calculation.

scalar function,

$$\mathcal{A}_n^{(L)} = \mathcal{A}_n^{\text{tree}} \mathcal{M}_n^{(L)}. \quad (2.1)$$

At one-loop order, the function $\mathcal{M}_n^{(1)}$ is expressed as a sum of so-called two-mass easy box functions $F^{2\text{me}}$ [11], all with coefficient equal to one:²

$$\mathcal{M}_n^{(1)} = \sum_{p,q} F^{2\text{me}}(p, q, P, Q). \quad (2.2)$$

The main characteristic of this function, depicted in Figure 1, is that two opposite legs, p and q , are massless, whereas the two remaining legs P and Q are massive. The summation in (2.2) is such that each different two-mass easy function appears exactly once.³

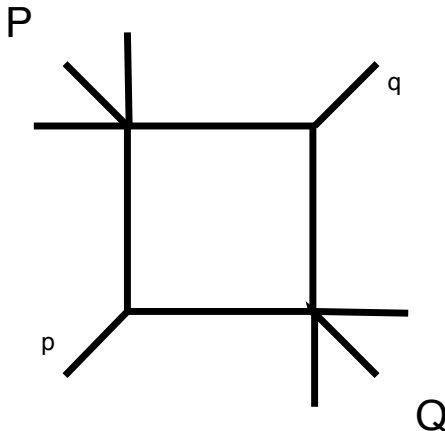


Figure 1: A two-mass easy box function. The momenta p and q are null, whereas, in general, $P^2 \neq 0$ and $Q^2 \neq 0$. The cases when either P^2 or Q^2 , or both, are also null, correspond to the one-mass and zero-mass boxes, obtained as smooth limits from the expression (2.3) of the two-mass box function.

A compact form of the two-mass easy box function containing only four dilogarithms was first derived in [12]. This form was found independently in [10] in the context of MHV diagrams, where an analytic proof of its equivalence with the conventional expression of e.g. [11] was given. Expressing the two-mass easy box as a function of the kinematic invariants $s := (P + p)^2$, $t := (P + q)^2$ and P^2 , Q^2 , with

²In (2.2) we are suppressing a factor of $c_\Gamma := \Gamma(1 + \epsilon)\Gamma^2(1 - \epsilon)/[(4\pi)^{2-\epsilon}\Gamma(1 - 2\epsilon)]$.

³A more explicit way to write $\mathcal{M}_n^{(1)}$ is $\mathcal{M}_n^{(1)} = \sum_{i=1}^n \sum_{r=1}^{\lfloor \frac{n}{2} \rfloor - 1} \left(1 - (1/2)\delta_{n/2-1,r}\right) F_{n:r;i}^{2\text{me}}$, where the relation to the functions introduced in (2.2) and depicted in Figure 1 is obtained by setting $p = p_{i-1}$, $q = p_{i+r}$, and $P = p_i + \dots + p_{i+r-1}$.

$p + q + P + Q = 0$, it reads

$$F^{2\text{me}}(s, t, P^2, Q^2) = -\frac{1}{\epsilon^2} \left[\left(\frac{-s}{\mu^2} \right)^{-\epsilon} + \left(\frac{-t}{\mu^2} \right)^{-\epsilon} - \left(\frac{-P^2}{\mu^2} \right)^{-\epsilon} - \left(\frac{-Q^2}{\mu^2} \right)^{-\epsilon} \right] \quad (2.3)$$

$$+ \text{Li}_2(1 - aP^2) + \text{Li}_2(1 - aQ^2) - \text{Li}_2(1 - as) - \text{Li}_2(1 - at) ,$$

where

$$a = \frac{P^2 + Q^2 - s - t}{P^2 Q^2 - st} = \frac{u}{P^2 Q^2 - st} . \quad (2.4)$$

For later use, we also quote the all-orders in ϵ expression of this function [13],

$$F^{2\text{me}}(s, t, P^2, Q^2) = -\frac{1}{\epsilon^2} \left[\left(\frac{-s}{\mu^2} \right)^{-\epsilon} + \left(\frac{-t}{\mu^2} \right)^{-\epsilon} - \left(\frac{-P^2}{\mu^2} \right)^{-\epsilon} - \left(\frac{-Q^2}{\mu^2} \right)^{-\epsilon} \right] \quad (2.5)$$

$$+ \left(\frac{a\mu^2}{1 - aP^2} \right)^\epsilon {}_2F_1 \left(\epsilon, \epsilon, 1 + \epsilon, \frac{1}{1 - aP^2} \right) + \left(\frac{a\mu^2}{1 - aQ^2} \right)^\epsilon {}_2F_1 \left(\epsilon, \epsilon, 1 + \epsilon, \frac{1}{1 - aQ^2} \right)$$

$$- \left(\frac{a\mu^2}{1 - as} \right)^\epsilon {}_2F_1 \left(\epsilon, \epsilon, 1 + \epsilon, \frac{1}{1 - as} \right) - \left(\frac{a\mu^2}{1 - at} \right)^\epsilon {}_2F_1 \left(\epsilon, \epsilon, 1 + \epsilon, \frac{1}{1 - at} \right) \Big] .$$

One important property of the MHV amplitudes is that they do not have multi-particle singularities. In particular, we note that, although each box function (2.3) contains poles in ϵ associated to multiparticle invariants, after performing the sum (2.2) the infrared divergent terms only involve two-particle invariants,

$$\mathcal{M}_n^{(1)}|_{\text{IR}} = -\frac{1}{\epsilon^2} \sum_{i=1}^n \left(\frac{-s_{ii+1}}{\mu^2} \right)^{-\epsilon} , \quad (2.6)$$

with $s_{ij} := (p_i + p_{i+1})^2$.

Recently, Bern, Dixon and Smirnov (BDS) proposed a remarkably simple conjecture for the resummation at all loops of the planar MHV amplitudes in $\mathcal{N}=4$ SYM calculated at weak coupling [1]. This conjecture, based on explicit calculations at two loops [14], was verified in [1] up to three loops in the four-point case, and in [15] up to two loops at five points. Explicit expressions of the four-point amplitudes at four and five loops were recently presented in [3] and [6], respectively, and will allow for precise tests of the conjecture at four and five loops once the relevant integral functions have been evaluated to the necessary degree of accuracy in ϵ .

The form of the BDS conjecture is inspired by the soft and collinear behaviour of amplitudes in gauge theory [16–23], and is expressed by

$$\mathcal{M}_n := 1 + \sum_{L=1}^{\infty} a^L \mathcal{M}_n^{(L)}(\epsilon) = \exp \left[\sum_{L=1}^{\infty} a^L \left(f^{(L)}(\epsilon) \mathcal{M}_n^{(1)} + C^{(L)} + E_n^{(L)}(\epsilon) \right) \right] , \quad (2.7)$$

where $a = [g^2 N / (8\pi^2)] (4\pi e^{-\gamma})^\epsilon$. Here $f^{(L)}(\epsilon) = f_0^{(L)} + f_1^{(L)}\epsilon + f_2^{(L)}\epsilon^2$ is a set of functions, one at each loop order, which make their appearance in the exponentiated all-loop expression for the infrared divergences in generic amplitudes in dimensional regularisation [21]. In particular, $f_0^{(L)} = \gamma_K^{(L)}/4$, where γ_K is the cusp anomalous dimension (related to the anomalous dimension of twist-two operators of large spin). An important point of the conjecture is that the constants $C^{(L)}$ do not depend on kinematics or on the number of particles n . The non-iterating contributions $E_n^{(L)}$ vanish as $\epsilon \rightarrow 0$ and depend explicitly on n .

BDS also propose a very interesting form for the finite remainders of the MHV amplitude, given by

$$\mathcal{F}_n = \exp \left[\frac{1}{4} \gamma_K F_n^{(1)}(0) + C \right], \quad (2.8)$$

where $F_n^{(1)}(0)$ is the finite remainder of $\mathcal{M}_n^{(1)}$ as defined in [1].

Motivated by the BDS conjecture, Alday and Maldacena have managed to reproduce the exponential formula (2.7) at strong coupling for the four-point case using the AdS/CFT correspondence [7]. The AdS dual description of a planar colour-ordered amplitude in $\mathcal{N}=4$ SYM is given by a classical open string worldsheet ending on a brane placed in the far infrared region of AdS space. Specifically, using coordinates in which the metric of AdS space is

$$ds^2 = R^2 \left(\frac{dx^2 + dz^2}{z^2} \right), \quad (2.9)$$

then the boundary of AdS space is at $z = 0$ and the infrared brane sits at $z = \infty$.

It is convenient however to use a T-dual description of this string configuration. This T-duality maps the AdS space into a new space with coordinates (y^μ, r) which has the metric

$$d\tilde{s}^2 = R^2 \left(\frac{dy^2 + dr^2}{r^2} \right), \quad (2.10)$$

where $r = R^2/z$. We see that the new space is again AdS, but now the infrared region and the boundary have been inverted. Therefore, the brane is located on the boundary in the new coordinates. Furthermore, the momenta of the particles p_i are expressed as differences of dual, or region momenta y_i [24], $(2\pi)p_i = y_i - y_{i+1}$. The calculation of the amplitude thus becomes that of finding the classical action S_{cl} of a string worldsheet whose boundary is a polygon with vertices y_i lying within the AdS boundary,

$$\mathcal{M}_n \sim e^{iS_{\text{cl}}}. \quad (2.11)$$

For the four-point amplitude, the corresponding string solution can be determined [7] giving $iS_{\text{cl}} = \text{div} + (\sqrt{\lambda}/8\pi) \log^2(s/t) + C$ where div represents divergent terms.

This agrees precisely with the BDS conjecture (2.8) and predicts $\gamma_K = \sqrt{\lambda}/4\pi$ at large $\lambda = g^2 N$, in agreement with string calculations [25, 26].

Now, the minimal area of a string ending on a path in the boundary of AdS space gives the vacuum expectation value of the Wilson loop over the same path in the CFT at strong coupling [27, 28]. A subtlety arising in the case at hand is the presence of singular points or cusps in the path, which lead to divergences [7, 29]. Nevertheless the divergences can be regularised by dimensional reduction even in the string calculation. Therefore, at least at strong coupling there is evidence for a dual description of amplitudes as Wilson loops. In the next section we will consider this possibility at small coupling. An important point to note here is that the string calculation does not depend on the species or helicities of the particles in the amplitude. These are subleading terms which would require α' corrections [7, 30]. Our Wilson loop calculation is also insensitive to the helicities of the scattered particles; thus, similarly to the Alday-Maldacena result, it does not generate the tree-level Parke-Taylor amplitude.

One mysterious and intriguing consequence of this dual description of amplitudes as Wilson loops is the unexpected appearance of conformal symmetry. Wilson loops of smooth paths in $\mathcal{N}=4$ SYM are conformally invariant objects (modulo an anomaly which does not depend either on the shape or size of the loop [31, 32]). However here the Wilson loop is divergent, since the path is not smooth, and regularisation spoils the conformal symmetry. Nevertheless, a similar pseudo-conformality seems to appear at weak coupling where all the integrals contributing to four-point MHV diagrams can be determined by rewriting them using the region momenta and appealing to off-shell conformality [3, 5, 8]. Furthermore at four points all conformal integrals of a certain type and with certain singular properties appear with coefficients ± 1 . The Wilson loop picture would seem to suggest that this pseudo-conformal invariance should continue for n -point functions. This point clearly deserves further investigation.

3 MHV amplitudes from a Wilson loop calculation

In this section we calculate one-loop corrections to the vacuum expectation value of a particular Wilson loop. In $\mathcal{N}=4$ SYM, the appropriate operator takes the form (suppressing fermions) [33–35]

$$W[\mathcal{C}] := \text{Tr } \mathcal{P} \exp \left[ig \oint_{\mathcal{C}} d\tau \left(A_{\mu}(x(\tau)) \dot{x}^{\mu}(\tau) + \phi_i(x(\tau)) \dot{y}^i(\tau) \right) \right], \quad (3.1)$$

where the ϕ_i 's are the six scalar fields of $\mathcal{N}=4$ SYM, and $(x^{\mu}(\tau), y^i(\tau))$ parametrise the loop \mathcal{C} . Importantly, when $\dot{x}^2 = \dot{y}^2$, the Wilson loop (3.1) is locally supersymmetric. The specific form of the contour \mathcal{C} we choose is dictated by the gluon momenta

p_1, \dots, p_n . Specifically, the segment associated to momentum p_i will be delimited by k_i and k_{i+1} ,

$$p_i := k_i - k_{i+1} , \quad (3.2)$$

and will be parametrised as $k_i(\tau_i) := k_i + \tau_i(k_{i+1} - k_i) = k_i - \tau_i p_i$, $\tau_i \in [0, 1]$. Momentum conservation $\sum_{i=1}^n p_i = 0$ implies that the contour is closed. In addition we set $\dot{y}^i = 0$ which makes the Wilson loop locally supersymmetric, as the gluon momenta, and hence the segments of the contour are null.⁴ However, we notice that each segment of the loop preserves a different subset of supersymmetries, therefore supersymmetry is broken globally.

We notice that the coordinates k_i can be interpreted as dual, or region momenta [24]. Indeed, for any planar diagrams one can express the momentum carried by a line as the difference of the momenta of the two regions of the plane separated by the segment. These coordinates have been used in the context of amplitude calculations by Thorn and collaborators in [36,37] and, more recently, in [38] in the context of MHV diagrams. They are also the T-dual coordinates introduced in [7] which determine the classical string solutions, as discussed earlier, and appear in the conformal integrals discussed in [5].

The four-particle case was recently addressed in [8], where it was found that the result of a one-loop Wilson loop calculation reproduces the four-point MHV amplitude in $\mathcal{N}=4$ SYM. Here we extend this result in two directions. First, we derive the four-point MHV amplitude to all-orders in the dimensional regularisation parameter ϵ . Secondly, we show that this striking agreement persists for an MHV amplitude with an arbitrary number of external particles.

Three different classes of diagrams give one-loop corrections to the Wilson loop.⁵ In the first one, a gluon stretches between points belonging to the same segment. It is immediately seen [8] that these diagrams give a vanishing contribution. In the second class of diagrams, a gluon stretches between two adjacent segments meeting at a cusp. Such diagrams are ultraviolet divergent and were calculated long ago [39–46], specifically in [45,46] for the case of gluons attached to lightlike segments.

In order to compute these diagrams, we will use the gluon propagator in the dual configuration space, which in $D = 4 - 2\epsilon_{UV}$ dimensions is

$$\begin{aligned} \Delta_{\mu\nu}(z) &:= -\frac{\pi^{2-\frac{D}{2}}}{4\pi^2} \Gamma\left(\frac{D}{2} - 1\right) \frac{\eta_{\mu\nu}}{(-z^2 + i\varepsilon)^{\frac{D}{2}-1}} \\ &= -\frac{\pi^{\epsilon_{UV}}}{4\pi^2} \Gamma(1 - \epsilon_{UV}) \frac{\eta_{\mu\nu}}{(-z^2 + i\varepsilon)^{1-\epsilon_{UV}}} . \end{aligned} \quad (3.3)$$

⁴We thank George Georgiou for discussions on this point.

⁵Notice that, for a Wilson loop bounded by gluons, we can only exchange gluons at one loop.

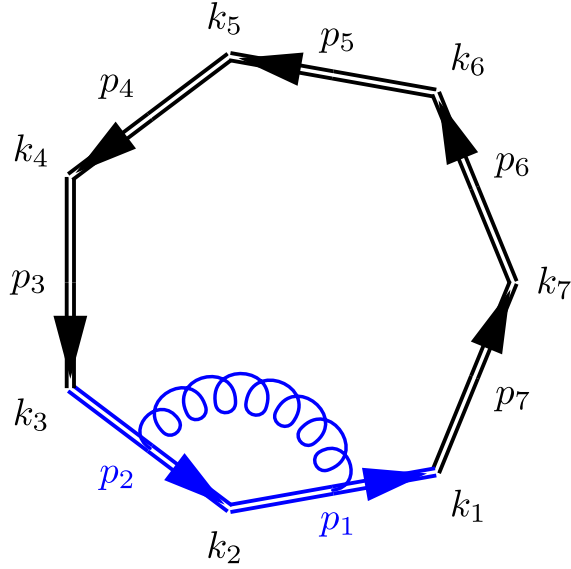


Figure 2: A one-loop correction to the Wilson loop, where the gluon stretches between two lightlike momenta meeting at a cusp. Diagrams in this class provide the infrared-divergent terms in the n -point scattering amplitudes, given in (2.6).

A typical diagram in the second class is pictured in Figure 2. There one has $k_1(\tau_1) - k_2(\tau_2) = p_1(1 - \tau_1) + p_2\tau_2$, where we used $p_1 = k_1 - k_2$ and $p_2 = k_2 - k_3$. The cusp diagram then gives⁶

$$\begin{aligned}
& -(ig\tilde{\mu}^{\epsilon_{UV}})^2 \frac{\Gamma(1 - \epsilon_{UV})}{4\pi^{2-\epsilon_{UV}}} \int_0^1 d\tau_1 d\tau_2 \frac{(p_1 p_2)}{[-(p_1 \tau_1 + p_2 \tau_2)^2]^{1-\epsilon_{UV}}} \\
& = -(ig\tilde{\mu}^{\epsilon_{UV}})^2 \frac{\Gamma(1 - \epsilon_{UV})}{4\pi^{2-\epsilon_{UV}}} \left[-\frac{1}{2} \frac{(-s)^{\epsilon_{UV}}}{\epsilon_{UV}^2} \right]. \tag{3.4}
\end{aligned}$$

The UV divergence should be interpreted as a divergence at small differences of region momenta, i.e. momenta, hence we interpret it as an infrared singularity in momentum space. Notice that $\epsilon_{UV} > 0$, in order to regulate the divergence in (3.4). Furthermore the scale used in the Wilson loop calculation is related to the scale used to regulate the amplitudes μ as $\tilde{\mu} = (c\mu)^{-1}$ (the precise coefficient c in front of μ can be reabsorbed into an appropriate redefinition of the coupling constant).

The last class of diagrams consists of diagrams where the gluon connects non-adjacent segments, such as that pictured in Figure 3. We denote by p and q the momenta carried by the two segments, and calculate the one-loop contribution due to the gluon exchange. We also set $p := k_p - k_{p+1}$, $q := k_q - k_{q+1}$. The gluon

⁶After changing variables $1 - \tau_1 \rightarrow \tau_1$.

propagator is a function of

$$(k_p(\tau_p) - k_q(\tau_q))^2 = \left(\sum_{i=p}^{q-1} (k_i - k_{i+1}) - \tau_p p + \tau_q q \right)^2 = \left(p(1 - \tau_p) + q\tau_q + \sum_{i=p+1}^{q-1} (k_i - k_{i+1}) \right)^2 \quad (3.5)$$

We recognise that $\sum_{i=p+1}^{q-1} (k_i - k_{i+1}) = P$ is the sum of the momenta between p and q , where, in general, $P^2 \neq 0$. Hence

$$\begin{aligned} (k_p(\tau_p) - k_q(\tau_q))^2 &= P^2 + 2(pP)(1 - \tau_p) + 2(qP)\tau_q + 2(pq)\tau_p\tau_q \\ &= P^2 + (s - P^2)(1 - \tau_p) + (t - P^2)\tau_q + (-s - t + P^2 + Q^2)\tau_p\tau_q, \end{aligned} \quad (3.6)$$

where we have re-expressed the result in terms of the invariants $s := (P + p)^2$, $t := (P + q)^2$ defined earlier. We can also introduce $u := -s - t + P^2 + Q^2$.

The one-loop diagram in Figure 3 is equal to

$$- (ig\tilde{\mu}^{\epsilon_{UV}})^2 \frac{1}{2} \frac{\Gamma(1 - \epsilon_{UV})}{4\pi^{2-\epsilon_{UV}}} \mathcal{F}_\epsilon(s, t, P, Q), \quad (3.7)$$

where $\mathcal{F}_\epsilon(s, t, P, Q)$ is the following two-dimensional integral,⁷

$$\mathcal{F}_\epsilon(s, t, P, Q) = \int_0^1 d\tau_p d\tau_q \frac{P^2 + Q^2 - s - t}{[-(P^2 + (s - P^2)\tau_p + (t - P^2)\tau_q + (-s - t + P^2 + Q^2)\tau_p\tau_q)]^{1+\epsilon}}. \quad (3.8)$$

The integral is finite in four dimensions. We begin by calculating it in four dimensions setting $\epsilon = 0$ (and will come back later to the calculation for $\epsilon \neq 0$). In this case, the result is

$$\begin{aligned} \mathcal{F}_{\epsilon=0}(s, t, P, Q) &= \text{Li}_2(as) + \text{Li}_2(at) - \text{Li}_2(aP^2) - \text{Li}_2(aQ^2) \\ &+ \log s \log \frac{(P^2 - s)(Q^2 - s)}{P^2Q^2 - st} + \log t \log \frac{(P^2 - t)(Q^2 - t)}{P^2Q^2 - st} \\ &- \log P^2 \log \frac{-(P^2 - s)(P^2 - t)}{P^2Q^2 - st} - \log Q^2 \log \frac{-(Q^2 - s)(Q^2 - t)}{P^2Q^2 - st}, \end{aligned} \quad (3.9)$$

where a is defined in (2.4). Using Euler's identity

$$\text{Li}_2(z) = -\text{Li}_2(1 - z) - \log z \log(1 - z) + \frac{\pi^2}{6}, \quad (3.10)$$

and noticing that [10] $(1 - as)(1 - at)/[(1 - aP^2)(1 - aQ^2)] = 1$, we can rewrite

$$\begin{aligned} \text{Li}_2(as) + \text{Li}_2(at) - \text{Li}_2(aP^2) - \text{Li}_2(aQ^2) &= \\ - \text{Li}_2(1 - as) - \text{Li}_2(1 - at) + \text{Li}_2(1 - aP^2) + \text{Li}_2(1 - aQ^2) \\ - \log s \log(1 - as) - \log t \log(1 - at) + \log P^2 \log(1 - aP^2) + \log Q^2 \log(1 - aQ^2). \end{aligned} \quad (3.11)$$

⁷In the following we set $\epsilon := -\epsilon_{UV} < 0$, where ϵ will correspond to the usual infrared regulator.

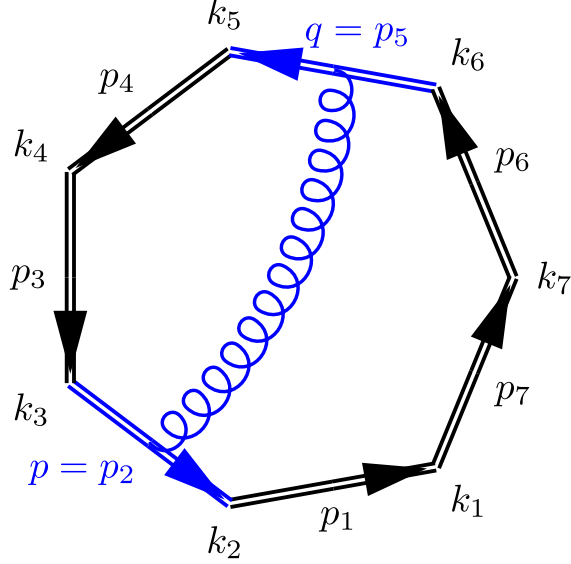


Figure 3: *Diagrams in this class – where a gluon connects two non-adjacent segments – are finite, and give a contribution equal to the finite part of a two-mass easy box function $F^{\text{2me}}(p, q, P, Q)$, second line of (2.3). p and q are the massless legs of the two-mass easy box, and correspond to the segments which are connected by the gluon. The diagram depends on the other gluon momenta only through the combinations P and Q .*

Upon making use of the relations [10]

$$\begin{aligned} 1 - as &= \frac{(s - P^2)(s - Q^2)}{P^2 Q^2 - st}, & 1 - at &= \frac{(t - P^2)(t - Q^2)}{P^2 Q^2 - st}, \\ 1 - aP^2 &= -\frac{(s - P^2)(t - P^2)}{P^2 Q^2 - st}, & 1 - aQ^2 &= -\frac{(s - Q^2)(t - Q^2)}{P^2 Q^2 - st}, \end{aligned} \quad (3.12)$$

we see that the terms in (3.9) containing logarithms cancel, and we are left with

$$\mathcal{F}_{\epsilon=0} = -\text{Li}_2(1 - as) - \text{Li}_2(1 - at) + \text{Li}_2(1 - aP^2) + \text{Li}_2(1 - aQ^2). \quad (3.13)$$

We conclude that $\mathcal{F}_{\epsilon=0}$ is precisely equal to the finite part of the two-mass easy box function – second line of (2.3). Finally, summing over all possible pairs of non-adjacent segments reproduces precisely the sum over box functions in (2.2).

In the four-point case, $a|_{P^2=Q^2=0} = 1/s + 1/t$, and using

$$\text{Li}_2(z) + \text{Li}_2\left(\frac{1}{z}\right) + \frac{1}{2} \log^2(-z) + \frac{\pi^2}{6} = 0, \quad (3.14)$$

one immediately finds [8]

$$\mathcal{F}_{\epsilon=0}|_{P^2=Q^2=0} = \frac{1}{2} \log^2\left(\frac{s}{t}\right) + \frac{\pi^2}{2}, \quad (3.15)$$

in complete agreement with the finite parts of the zero-mass box function (in the normalisations of [9]).

Finally, we discuss the calculation at $\epsilon \neq 0$. In this case one finds that

$$\mathcal{F}_\epsilon = -\frac{1}{\epsilon^2} \quad (3.16)$$

$$\cdot \left[\left(\frac{a}{1-aP^2} \right)^\epsilon {}_2F_1 \left(\epsilon, \epsilon, 1+\epsilon, \frac{1}{1-aP^2} \right) + \left(\frac{a}{1-aQ^2} \right)^\epsilon {}_2F_1 \left(\epsilon, \epsilon, 1+\epsilon, \frac{1}{1-aQ^2} \right) \right.$$

$$\left. - \left(\frac{a}{1-as} \right)^\epsilon {}_2F_1 \left(\epsilon, \epsilon, 1+\epsilon, \frac{1}{1-as} \right) - \left(\frac{a}{1-at} \right)^\epsilon {}_2F_1 \left(\epsilon, \epsilon, 1+\epsilon, \frac{1}{1-at} \right) \right] .$$

This result is in precise agreement with the finite part of the all-orders in ϵ two-mass easy box function, second and third line of (2.5). Notice that the expression in (3.16) is finite as $\epsilon \rightarrow 0$.

For $n > 4$, simply replacing the $\epsilon = 0$ expression of the box functions with their all-orders in ϵ expression does not provide us with a complete, all-orders in ϵ expression for the amplitude, as there are additional n -gon integrals which vanish as $\epsilon \rightarrow 0$, which are not included. The four-point case is an exception, and in this case our calculation reproduces the expected all-orders in ϵ result. Combining the infrared-divergent and finite terms, our result is

$$\mathcal{M}_4^{(1)}(\epsilon) = -\frac{2}{\epsilon^2} \left[\left(\frac{-s}{\mu^2} \right)^{-\epsilon} {}_2F_1 \left(1, -\epsilon, 1-\epsilon, 1+\frac{s}{t} \right) + \left(\frac{-t}{\mu^2} \right)^{-\epsilon} {}_2F_1 \left(1, -\epsilon, 1-\epsilon, 1+\frac{t}{s} \right) \right], \quad (3.17)$$

in agreement with [47].

Finally, we would like to mention that it is easy to see that splitting amplitudes and soft functions can also be derived at one loop using Wilson loops. Furthermore, they have an interesting interpretation in terms of the geometry of the contour; for instance, splitting amplitudes arise from two adjacent segments becoming nearly parallel to each other, therefore merging into a single segment.

4 Conclusions

We found that a strikingly simple one-loop gluon exchange calculation of a Wilson loop, whose (closed) boundary is defined by a set of lightlike segments p_1, \dots, p_n , reproduces the n -point one-loop MHV amplitudes in $\mathcal{N} = 4$ SYM (divided by the tree-level amplitude). We would like to comment on this surprising result.

1. One of the important features of the calculation presented in this paper is that it neatly separates the infrared-divergent terms from the finite parts. The latter can then be derived working directly in four dimensions, which turns out to be a key calculational advantage. For this reason, we are hopeful that this procedure could allow for a direct check of the exponentiation of the finite remainders [1]. The perspective of deriving a field-theoretical proof of the all-loop expressions of [1] from Wilson loops – possibly using the non-abelian exponentiation theorem [48, 49] – is an exciting one.

2. In [10], it was speculated that the two-mass easy box functions should emerge naturally as Feynman diagrams of the perturbative description of a (possibly string) theory. This was motivated by the observation that the MHV amplitudes in $\mathcal{N}=4$ SYM at one loop are written as sum of box functions, each appearing with coefficient one. In this paper, we have found that the Wilson loop calculation gives a precise, one-to-one mapping of Wilson loop diagrams to the *finite part* of two-mass easy box functions (or, in specific cases, the degenerate one-mass and zero-mass functions). The massless legs of the box function, p and q , are simply those to which the gluon is attached (see Figure 3). The calculation is only sensitive to p , q , and the *sum* P of the momenta between p and q . Therefore, the Wilson loop calculation seems to have provided such a description where a two-mass easy box is a specific Wilson loop diagram (notice that for the diagram with a gluon connecting segments p and q , the remaining part of the loop could be deformed to the shape of a (generically) two-mass box function).⁸ It is tempting to speculate that the observation of [3] that even at higher loops the MHV amplitudes are expressed as sums of (conformal) integrals, each appearing with coefficients ± 1 , could be explained in terms of higher-loop Wilson loop diagrams.

3. We believe that the agreement found in this paper between our Wilson loop calculation and the $\mathcal{N}=4$ MHV amplitude with an arbitrary number of points lends support to the conjecture that the appropriate strongly-coupled string theory calculation at n points will confirm the BDS conjecture for the full exponentiated expression of the n -point MHV amplitude.

⁸We also note in passing that we have found a representation of the finite part of two-mass easy box functions in terms of a very simple two-dimensional integral, see (3.8) and (3.13).

Acknowledgements

It is a pleasure to thank James Drummond, George Georgiou, Valeria Gili, Gregory Korchemsky, Sanjaye Ramgoolam, Rodolfo Russo, Emery Sokatchev, Bill Spence and Costas Zoubos for discussions. We would like to thank PPARC for support under a Rolling Grant PP/D507323/1 and the Special Programme Grant PP/C50426X/1. The work of PH is supported by an EPSRC Standard Research Grant EP/C544250/1. GT is supported by an EPSRC Advanced Fellowship EP/C544242/1 and by an EPSRC Standard Research Grant EP/C544250/1.

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